On S-permutable Subgroups of Finite Groups

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1 Introduction

Recall that two subgroups A and B of a group G are said to permute if $AB = BA$. It is easily seen that A and B permute iff the set AB is a subgroup of G. A subgroup A of the group G is called quasinormal [1] or permutable [2, 3] in G if it permutes with all subgroups of G.

[1] O.Ore, Contributions in the theory of groups of finite order, Duke Math. J., **5**, 1939, 431–460.

[2] S.E.Stonehewer, Permutable subgroups in Infinite Groups, Math. Z., **125**, 1972, 1–16.

[3] K.Doerk and T.Hawkes, "Finite soluble grousp", Walter de gruyter, Berlin/New York, 1992.

The permutable subgroups have many interesting properties especially in the case when G is a finite group. It was observed by Ore [1] that every permutable subgroup H of a finite group G is subnormal.

By extending this result, Ito and Szĕp have proved in [4]

[4] N.Ito and J,Szép, Uber die Quasinormalteiler von endlichen Gruppen, Act. Sci. Math., **23**, 1962, 168–170.

that for every permutable subgroup A of a finite group G, A/A_G is nilpotent. Here A_G is the kernel of A, that is the largest normal subgroup of G contained in A.

Another important result related to Ore's result was obtained by Stonehewer in [2] in which he has proved that every permutable subgroup of every finitely generated group G is subnormal in G.

Some later, Maier and Schmid proved in [5]

[5] Maier R., Schmid P. The embedding of permutable subgroups in finite groups, Math. Z., **131**, 1973, 269–272.

that for every permutable subgroup A of G it is true that $A^G/A_G \subseteq Z_\infty(G/A_G)$. Here A^G is the normal closure of A in G that is the intersection of all such normal subgroups of G which contain A. This result shows that "difference" between normality and permutability in general is small and several authors have investigated subgroups of finite groups which are permutable with all subgroups of some given system of subgroups. In this connection we first of all have to remind here about the following paper by Kegel

[6] O.H.Kegel. Sylow-Gruppen and Subnormalteiler endlicher Gruppen, Math. Z., **87**, 1962, 205–221.

A subgroup A of a group G is called s-quasinormal or s-permutable or π-permutable if it permutes with all Sylow subgroups of G.

It was discovered by Kegel [5] and by Deskins in [W.E.Deskins, On quasinormal subgroups of finite groups, Math. Z., **82**, 1963, 125–132] that subgroups of this kind have the properties similar to the properties of permutable subgroups and, in particular, they are subnormal.

After these two papers several authors were studing and applying s-permutable subgroups. My main goal here is to discuss some new applications od such subgroups.

2 Results

Several authors have investigated the structure of a group G under the assumption that the maximal or the minimal subgroups of the Sylow subgroups of some subgroups of G are well situated in G. Buckly in [7]

[7] Buckley J. Finite groups whose minimal subgroups are normal, Math. Z., (15) 1970, 15–17.

proved that a group of odd order is supersoluble if all its minimal subgroups are normal. Later on, Srinivasan in [8]

[8] Srinivasan S. Two sufficient conditions for supersolubility of finite groups, Israel J. Math., 3(35) 1980, 210-214.

showed that a group G is supersoluble if it has a normal subgroup N with supersoluble quotient G/N such that all maximal subgroups of the Sylow subgroups of N are normal in G. Ramadan proved in [9]:

[9] Ramadan M. Influence of normality on maximal subgroups of Sylow subgroups of a finite group, Acta Math. Hungar., 59(1-2) 1992, 107-110.

If G is a soluble group and all maximal subgroups of any Sylow subgroup of $F(G)$ are normal in G, then G is supersoluble. Some later several authors were studing groups G in which the maximal or the minimal subgroups of the Sylow subgroups of some subgroups of G are s-permutable in G.

We mention here, for example, the following papers in this trend:

[10] Shaalan A. The influence of s-permutableity of some subgroups, Acta. Math. Hungar., (56) 1990, 287-293.

[11] Asaad M., Ramadan M., Shaalan A. Influence of s-permutableity on maximal subgroups of Sylow subgroupsof Fitting subgroups of a finite groups, Arch. Math., (56) 1991, 521-527.

[12] Asaad M. On the solvability of finite groups, Arch. Math., (51) 1988, 289-293.

[13] Asaad M., Csorgo P. Influence of minimal subgroups on the structure of finite group, Arch. Math., (72) 1999, 401-404.

[14] Asaad M. On maximal subgroups of finite group, Comm. Algebra, 26(11) 3647-3652, 1998.

[15] Li Y., Wang Y. The influence of s-permutableity of some subgroups of a finite group, Proc. Amer. Math. Soc. 131(2) 337-341, 2002.

[16] Li Y., Wang Y. The influence of s-permutability of some subgroups of a finite group, Arch. Math. 81 (2003), 245-252.

[17] Ballester-Bolinches A., Pedraza-Aguilera M.C. On minimal subgroups of finite groups, Acta Math. Hungar., 4(73) 1996, 121-127.

The most general results in this trend were obtained in [15, 16] where the following two nice theorems were proved:

Therem A. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G be a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F^*(E)$ are s-permutable G, then $G \in \mathcal{F}$* (see [16, Theorem 3.1).)

Theorem B *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G be a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If all maximal subgroups of the Sylow subgroups of $F^*(E)$ are s-permutable in G, then $G \in \mathcal{F}$* (see [15, Theorem 3.1]);

In these two theorems $F^*(E)$ is the generalized Fitting subgroup of E that is the product of all normal quasinilpotent subgroups of E. By Theorem 13.6 in [18, X]

[18] Huppert B., Blackburn N. Finite Groups III,, Berlin, New-York, Springer-Verlag, 1982.

a group G is quasinilpotent if and only if $G/Z_\infty(G)$ is the either identity group of the direct product of some simple non-abelian groups.

Recall that a formation $\mathfrak{F}$ is a class of groups which is closed under taking homomorphic images and such that each group G has a smallest normal subgroup with quotient in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-residual of G and it is denoted by $G^{\mathfrak{F}}$. A formation $\mathfrak{F}$ is called saturated if $\mathfrak{F}$ contains every group G with $G/\Phi(G)) \in \mathfrak{F}$.

In the connection with Theorems A, B the following natural question arises: Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Is the group G in $\mathcal{F}$ if for every Sylow subgroup P of $F^*(G)$ at least one of the following conditions holds:

(1) The maximal subgroups of P are s-permutable in G;

(2) The minimal subgroups of P and all its cyclic subgroups with order 4 are s-permutable in G?

We prove the following theorem which gives the positive answer to this question.

Theorem 1. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G be a group* *with a normal subgroup E such that $G/E \in \mathfrak{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are s-permutable in G. Then $G \in \mathfrak{F}$.*

One of the main steps in the proof of Theorem 1 is the following result.

Theorem 2. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G be a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) not having a supersoluble supplement in G are s-permutable in G. Then $G \in \mathfrak{F}$.*

3 Some applications

Several known results are special cases of Theorems 1 and 2. I want to mention in my talk about some of such results.

Corollary 1.1 (Buckley [7]). *Let G be a group of odd order. If all subgroups of G of prime order are normal in G, then G is supersoluble.*

[19] W.Guo, Shum K.P., Skiba A.N. G-covering subgroup systems for the classes of supersoluble and nilpotent groups, Israel J. of Math., (138) 2003, 125-138.

In [19] the following fact was proved

Corollary 1.2 (Guo W., Shum K.P. and Skiba A.N. [19]). *If the maximal subgroups of the Sylow subgroups of G not having supersoluble supplement in G are normal in G, then G is supersoluble.*

Corollary 1.3 (Srinivasan [8]). *If the maximal subgroups of the Sylow subgroups of G are s-permutable in G, then G is supersoluble.*

Corollary 1.4 (Shaalan A. [10]). *Let G be a group and E a normal subgroup of G with supersoluble quotient. Suppose that all minimal subgroups of E and all its cyclic subgroups with order 4 are s-permutable in G. Then G is supersoluble.*

Corollary 1.5 (Ballester-Bolinches A., Pedraza-Aguilera M.C. [17]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with normal subgroup E such that $G/E \in \mathfrak{F}$. Assume that a Sylow 2-subgroup of G is abelian. If all minimal subgroups of E are permutable in G, then $G \in \mathfrak{F}$.*

Corollary 1.6 (Ballester-Bolinches A., Pedraza-Aguilera M.C. [17]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with a soluble normal subgroup E such that $G/E \in \mathfrak{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of E are s-permutable in G, then $G \in \mathfrak{F}$.*

[20] Ramadan M. Influence of normality on maximal subgroups of Sylow subgroups of a finite group, Acta Math. Hungar., 59(1-2) 1992, 107-110.

Corollary 2.1 (Ramadan M. [9]). *Let G be a soluble group. If all maximal subgroups of the Sylow subgroups of $F(E)$ are normal in G, then G is supersoluble.*

Corollary 2.2 (Asaad M., Ramadan M. and Shaalan A. [11]). *Let G be a group and E a soluble normal subgroup of G with supersoluble quotient G/E. Suppose that all maximal subgroups of any Sylow subgroup of $F(E)$ are s-permutable in G. Then G is supersoluble.*

Corollary 2.3 (Asaad M., Csorgo P. [13]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G be a group with a soluble normal subgroup E such that $G/E \in \mathfrak{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(E)$ are s-permutable in G, then $G \in \mathfrak{F}$.*

Corollary 2.4 (Li Y., Wang Y. [15]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F^*(E)$ are s-permutable in G, then $G \in \mathfrak{F}$.*

Corollary 2.6 (Li Y., Wang Y. [16]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If all maximal subgroups of $F^*(E)$ are s-permutable in G, then $G \in \mathfrak{F}$.*