Linearization of self-semilar groups by dilatation structures

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In this article we study dilatation structures on the boundary of the dyadic tree. We answer then to the question: given a self-similar group of isometries of the dyadic tree, is there any dilatation structure which makes the group linear? Here linearity is meant in the generalized sense of dilatation structures: a transformation is linear if it commutes (in the right way) with dilatations.

We prove that the first Grigorchuk group is non-linear even in this generalized sense. On the countrary, the basilica group is generalized linear, even if it is not linear in the usual sense. Some iterated monodromy groups are then studied. We discuss by way of examples the possibility that a variant of Tits alternative might hold for self-similar groups which are generalized linear.

1 Notations

Let Γ be a topological separated commutative group endowed with a continuous group morphism

$$\nu : \Gamma \rightarrow (0, +\infty)$$

with $\inf \nu(\Gamma) = 0$. Here $(0, +\infty)$ is taken as a group with multiplication. The neutral element of Γ is denoted by 1. We use the multiplicative notation for the operation in Γ.

The morphism ν defines an invariant topological filter on Γ (equivalently, an end). Indeed, this is the filter generated by the open sets $\nu^{-1}(0, a)$, $a > 0$. From now on we shall name this topological filter (end) by "0" and we shall write $\varepsilon \in \Gamma \rightarrow 0$ for $\nu(\varepsilon) \in (0, +\infty) \rightarrow 0$.

The set $\Gamma_1 = \nu^{-1}(0, 1]$ is a semigroup. We note $\bar{\Gamma}_1 = \Gamma_1 \cup \{0\}$ On the set $\bar{\Gamma} = \Gamma \cup \{0\}$ we extend the operation on Γ by adding the rules $00 = 0$ and $\varepsilon 0 = 0$ for any $\varepsilon \in \Gamma$. This is in agreement with the invariance of the end 0 with respect to translations in Γ.

We shall use the following convenient notation: by $\mathcal{O}(\varepsilon)$ we mean a positive function defined on Γ such that $\lim_{\varepsilon \rightarrow 0} \mathcal{O}(\nu(\varepsilon)) = 0$.

1.1 Words and the Cantor middle-thirds set

Let X be a finite, non empty set. The elements of X are called letters. The collection of words of finite length in the alphabet X is denoted by X^*. The empty word $\emptyset$ is an element of X^*.

The length of any word $w \in X^*$

$$w = a_1 ... a_m \quad , \quad a_k \in X \quad \forall k = 1, ..., m$$

is denoted by $\mid w \mid = m$.

The set of words which are infinite at right is denoted by

$$X^\omega = \{ f \mid f : \mathbb{N}^* \rightarrow X \} = X^{\mathbb{N}^*} \quad .$$

Concatenation of words is naturally defined. If $q_1, q_2 \in X^*$ and $w \in X^\omega$ then $q_1 q_2 \in X^*$ and $q_1 w \in X^\omega$.

The shift map $s : X^\omega \rightarrow X^\omega$ is defined by

$$w = w_1 \, s(w)$$

for any word $w \in X^\omega$. For any $k \in \mathbb{N}^*$ we define $[w]_k \in X^k \subset X^*$, $\{w\}_k \in X^\omega$ by

$$w = [w]_k \, s^k(w) \quad , \quad \{w\}_k = s^k(w) \quad .$$

The topology on X^ω is generated by cylindrical sets qX^ω, for all $q \in X^*$. The topological space X^ω is compact.

To any $q \in X^*$ is associated a continuous injective transformation $\hat{q} : X^\omega \rightarrow X^\omega$, $\hat{q}(w) = qw$. The semigroup X^* (with respect to concatenation) can be identified with the semigroup (with respect to function composition) of these transformations. This semigroup is obviously generated by X. The empty word corresponds to the identity function.

The dyadic tree $\mathcal{T}$ is the infinite rooted planar binary tree, with any node having two descendants. The nodes are coded by elements of X^*, $X = \{0, 1\}$. The root is coded by the empty word and if a node is coded by $x \in X^*$ then its left hand side descendant has the code $x0$ and its right hand side descendant has the code $x1$. We shall therefore identify the dyadic tree with X^* and we put on the dyadic tree the natural (ultrametric) distance on X^*. The boundary (or the set of ends) of the dyadic tree is then the same as the compact ultrametric space X^ω.

1.2 Isometries of the dyadic tree

An isomorphism of $\mathcal{T}$ is just an invertible transformation which preserves the structure of the tree. It is well known that isometries of (X^ω, d) are the same as isometries of $\mathcal{T}$.

Let $A \in Isom(X^\omega, d)$ be such an isometry. For any finite word $q \in X^*$ we may define $A_q \in Isom(X^\omega, d)$ by

$$A(qw) = A(q) \, A_q(w)$$

for any $w \in X^\omega$. Note that in the previous relation $A(q)$ makes sense because A is also an isometry of $\mathcal{T}$.

Definition 1.1 *A group $G \subset Isom(X^\omega)$ is self-similar if for any $q \in X^*$ and any $A \in G$ we have $A_q \in G$.*

Let G be a finitely generated self-similar group which has a generating set $M = \{a_i : i = 1, ..., n\}$ such that :

(a) it contains the identity ,

(b) for any $i \in \{1, ..., n\}$ and $\alpha \in X$ there is $j = h(i, \alpha) \in \{1, ..., n\}$ such that for any $w \in X^{\omega}$ we have $a_i(\alpha w) = g(i, \alpha) a_j(w)$.

Then we associate to M the Moore diagram with the set of nodes M and the set of arrows containing all pairs (a_i, a_j) with $j = h(i, \alpha)$, $\alpha \in X$. Any such arrow is marked with the pair $(\alpha, g(i, \alpha))$.

To any isometry $A \in Isom(X^{\omega})$ we associate a decoration of the dyadic tree obtained by marking the node $q \in X^*$ with A_q. The transformation A is of finite type if the set of all A_q, $q \in X^*$, is finite.

To any A of finite type we associate the group generated by all A_q, $q \in X^*$. The Moore diagram of this group (with respect to these generators) is connected, in the sense that there is an oriented path in the diagram which links any two generators. Conversely, if the Moore diagram is connected then there exists an isometry of finite type A which generates the group in the sense previously explained.

2 Motivation: linear structure in terms of dilatations

Linearity is a basic property related to vector spaces. If $\mathbb{V}$ is a real vector vector space, for example, then a transformation $A : \mathbb{V} \rightarrow \mathbb{V}$ is linear if it is a morphism of groups $A : (\mathbb{V}, +) \rightarrow (\mathbb{V}, +)$ and homogeneous with respect to positive scalars. Furthermore, in a normed vector space we can speak about linear continuous transformations. A group is then linear if it admits a faithful linear representation.

A transformation is affine if it is a composition of a translation with a linear transformation. In this paper we shall use the umbrella name "linear" for affine transformations too. This makes no difference concerning the definition of a linear group, because a group which admits a faithful affine representation is linear.

We try here to explain that linearity property can be entirely phrased in terms of dilatations of the vector space $\mathbb{V}$.

For the vector space $\mathbb{V}$, the dilatation based at x, of coefficient $\varepsilon > 0$, is the function

$$\delta^x_\varepsilon : \mathbb{V} \rightarrow \mathbb{V} \quad , \quad \delta^x_\varepsilon y = x + \varepsilon(-x + y) \quad .$$

For fixed x the dilatations based at x form a one parameter group which contracts any bounded neighbourhood of x to a point, uniformly with respect to x.

The algebraic structure of $\mathbb{V}$ is encoded in dilatations. Indeed, using dilatations we can recover the operation of addition and multiplication by scalars.

For $x, u, v \in \mathbb{V}$ and $\varepsilon > 0$ define

$$\Delta^x_\varepsilon(u, v) = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v \quad , \quad \Sigma^x_\varepsilon(u, v) = \delta^x_{\varepsilon^{-1}} \delta^{\delta^x_\varepsilon u}_\varepsilon (v) \quad , \quad inv^x_\varepsilon(u) = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} x \quad .$$

The meaning of this functions becomes clear if we compute:

$$\Delta^x_\varepsilon(u, v) = x + \varepsilon(-x + u) + (-u + v) \quad ,$$

$$\Sigma^x_\varepsilon(u, v) = u + \varepsilon(-u + x) + (-x + v) \quad ,$$

$$inv_{\varepsilon}^{x}(u) == x + \varepsilon(-x+u) + (-u+x) \quad .$$

As $\varepsilon \to 0$ we have the limits:

$$\lim_{\varepsilon\to 0} \Delta_{\varepsilon}^{x}(u,v) = \Delta^{x}(u,v) = x + (-u+v) \quad ,$$

$$\lim_{\varepsilon\to 0} \Sigma_{\varepsilon}^{x}(u,v) = \Sigma^{x}(u,v) = u + (-x+v) \quad ,$$

$$\lim_{\varepsilon\to 0} inv_{\varepsilon}^{x}(u) = inv^{x}(u) = x - u + x \quad ,$$

uniform with respect to x, u, v in bounded sets. The function $\Sigma^{x}(\cdot,\cdot)$ is a group operation, namely the addition operation translated such that the neutral element is x. Thus, for $x = 0$, we recover the group operation. The function $inv^{x}(\cdot)$ is the inverse function, and $\Delta^{x}(\cdot,\cdot)$ is the difference function.

Notice that for fixed x, ε the function $\Sigma_{\varepsilon}^{x}(\cdot,\cdot)$ is not a group operation, first of all because it is not associative. Nevertheless, this function satisfies a shifted associativity property, namely (see theorem 5.3)

$$\Sigma_{\varepsilon}^{x}(\Sigma_{\varepsilon}^{x}(u,v),w) = \Sigma_{\varepsilon}^{x}(u, \Sigma_{\varepsilon}^{\delta_{\varepsilon}^{x}u}(v,w)) \quad .$$

Also, the inverse function inv_{ε}^{x} is not involutive, but shifted involutive (theorem 5.3):

$$inv_{\varepsilon}^{\delta_{\varepsilon}^{x}u}(inv_{\varepsilon}^{x}u) = u \quad .$$

Dilatations behave well with respect to the distance d induced by the norm, in the following sense: for any $x, u, v \in \mathbb{V}$ and any $\varepsilon > 0$ we have

$$\frac{1}{\varepsilon} d(\delta_{\varepsilon}^{x}u, \delta_{\varepsilon}^{x}v) = d(u,v) \quad .$$

This shows that from the metric point of view the space $(\mathbb{V}, d)$ is a metric cone, that is $(\mathbb{V}, d)$ looks the same at all scales.

Affine continuous transformations $A : \mathbb{V} \to \mathbb{V}$ admit the following description in terms of dilatations. (We could dispense of continuity hypothesis in this situation, but we want to illustrate a general point of view, described further in the paper).

Proposition 2.1 *A continuous transformation* $A : \mathbb{V} \to \mathbb{V}$ *is affine if and only if for any* $\varepsilon \in (0,1)$*,* $x, y \in \mathbb{V}$ *we have*

$$A\delta_{\varepsilon}^{x}y \ = \ \delta_{\varepsilon}^{Ax}Ay \quad . \tag{2.0.1}$$

The proof is a straightforward consequence of representation formulæ for the addition, difference and inverse operations in terms of dilatations.

Let us go further with a more complex example. Consider the Heisenberg group $H(n)$. AS a set $H(1) = \mathbb{R}^{2n} \times \mathbb{R}$. We shall use the following notation: an element of $H(n)$ will be denoted by $\tilde{x} = (x, \bar{x})$, with $x \in \mathbb{R}^{2n}, \bar{x} \in \mathbb{R}$. The group operation is

$$\tilde{x}\tilde{y} = (x+y, \bar{x} + \bar{y} + 2\omega(x,y)) \quad ,$$

where ω is the canonic symplectic 2-form on $\mathbb{R}^{2n}$.

The group $H(n)$ is nilpotent, in fact a 2 graded Carnot group. This means that $H(n)$ is nilpotent and that it admits a one-parameter group of isomorphisms

$$\delta_\varepsilon(x, \bar{x}) = (\varepsilon x, \varepsilon^2 \bar{x}) \quad .$$

These are dilatations, more precisely we can construct dilatations based at $\tilde{x}$ by the formula

$$\delta_\varepsilon^{\tilde{x}} \tilde{u} = \tilde{x} \delta_\varepsilon \left(\tilde{x}^{-1} \tilde{u} \right) \quad .$$

We may also put a scalling invariant distance on $H(n)$, for example this one:

$$d(\tilde{x}, \tilde{y}) = g(\tilde{x}^{-1} \tilde{y}) \quad , \quad g(\tilde{u}) = \max \left\{ \|u\|, \sqrt{\mid \bar{u} \mid} \right\} \quad .$$

We can repeat step by step the constructions explained before in this situation. There are some differences though.

For example, from the metric point of view, $(H(n), d)$ is a fractal space, in the sense that the Hausdorff dimension of this space is equal to $2n+2$, therefore strictly greater than the topological dimension, which is $2n+1$.

The Heisenberg group is not commutative. It is in fact the model for the tangent space of a contact metric manifold, as the euclidean $\mathbb{R}^n$ is the model of the tangent space of a riemannian manifold.

Linear, continuous transformations of $H(n)$ in the sense of relation (2.0.1) turn out to be just compositions of left translations in $H(n)$ with group endomorphisms of $H(n)$ which commutes with dilatations δ_ε.

Dilatation structures were initially introduced in connection to sub-riemannian geometry. In that frame a dilatation structure has an associated differential calculus, generalizing the one using the Pansu derivative on Carnot groups [10].

Here we apply dilatation structures for ultrametric spaces. At this stage we shall not study the differential calculus associated to a dilatation structure on a ultrametric space.

We shall concentrate on dilatation structures on the boundary of the dyadic tree and the related notions of linearity of a self-similar group, in the sense of (2.0.1). We shall use therefore a notion of linearity (hence linear group) more general than usual.

3 Dilatation structures

This section contains notions and results introduced or proved in Buliga [4], [5]. In order to make the paper more self contained we add an appendix with further details about dilatation structures.

3.1 Axioms of dilatation structures

The axioms of a dilatation structure (X, d, δ) are listed further. The space (X, d) is a complete, locally compact metric space.

The first axiom is merely a preparation for the next axioms. That is why we counted it as axiom 0.

A0. The dilatations

$$\delta_\varepsilon^x : U(x) \to V_\varepsilon(x)$$

are defined for any $\varepsilon \in \Gamma, \nu(\varepsilon) \leq 1$. All dilatations are homeomorphisms (invertible, continuous, with continuous inverse).

We suppose that there is $1 < A$ such that for any $x \in X$ we have

$$\bar{B}_d(x, A) \subset U(x) \ .$$

We suppose that for all $\varepsilon \in \Gamma$, $\nu(\varepsilon) \in (0, 1)$, we have

$$B_d(x, \varepsilon) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset U(x) \ .$$

For $\nu(\varepsilon) \in (1, +\infty)$ the associated dilatation

$$\delta_\varepsilon^x : W_\varepsilon(x) \to B_d(x, B) \ ,$$

is injective, invertible on the image. We shall suppose that $W_\varepsilon(x)$ is open,

$$V_{\varepsilon^{-1}}(x) \subset W_\varepsilon(x)$$

and that for all $\varepsilon \in \Gamma_1$ and $u \in U(x)$ we have

$$\delta_{\varepsilon^{-1}}^x \ \delta_\varepsilon^x u \ = \ u \ .$$

A further technical condition on the sets $V_\varepsilon(x)$ and $W_\varepsilon(x)$ will be given just before the axiom A4. (This condition will be counted as part of axiom A0.)

Remark 3.1 *We deduce the following string of inclusions, for any $\varepsilon \in \Gamma$, $\nu(\varepsilon) \leq 1$, and any $x \in X$:*

$$B_d(x, \varepsilon) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta_\varepsilon^x B_d(x, B) \quad .$$

A1. We have $\delta_\varepsilon^x x = x$ for any point x. We also have $\delta_1^x = id$ for any $x \in X$.

Let us define the topological space

$$dom\, \delta = \{(\varepsilon, x, y) \in \Gamma \times X \times X : \quad \text{if } \nu(\varepsilon) \leq 1 \text{ then } y \in U(x) \ , \text{ else } y \in W_\varepsilon(x)\} \quad ,$$

with the topology inherited from the product topology on $\Gamma \times X \times X$. Consider also $Cl(dom\, \delta)$, the closure of $dom\, \delta$ in $\bar{\Gamma} \times X \times X$ with product topology. The function

$$\delta : dom\, \delta \to X$$

defined by $\delta(\varepsilon, x, y) = \delta_\varepsilon^x y$ is continuous. Moreover, it can be continuously extended to $Cl(dom\, \delta)$ and we have

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^x y = x \quad .$$

A2. For any $x, \in K$, $\varepsilon, \mu \in \Gamma_1$ and $u \in \bar{B}_d(x, A)$ we have:

$$\delta_\varepsilon^x \delta_\mu^x u = \delta_{\varepsilon\mu}^x u \ .$$

A3. For any x there is a function $(u, v) \mapsto d^x(u, v)$, defined for any u, v in the closed ball (in distance d) $\bar{B}(x, A)$, such that

$$\lim_{\varepsilon \rightarrow 0} \quad \sup \left\{ \mid \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ - \ d^x(u, v) \mid \ : \ u, v \in \bar{B}_d(x, A) \right\} \ = \ 0$$

uniformly with respect to x in compact set.

Remark 3.2 *The "distance" d^x can be degenerated. That means: there might be $v, w \in \bar{B}_d(x, A)$ such that $d^x(v, w) = 0$ but $v \neq w$. We shall use further the name "distance" for d^x, essentially by commodity, but keep in mind the possible degeneracy of d^x.*

For the following axiom to make sense we impose a technical condition on the co-domains $V_\varepsilon(x)$: for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0, 1)$ such that for all $u, v \in \bar{B}_d(x, R)$ and all $\varepsilon \in \Gamma$, $\nu(\varepsilon) \in (0, \varepsilon_0)$, we have

$$\delta_\varepsilon^x v \in W_{\varepsilon^{-1}}(\delta_\varepsilon^x u) \ .$$

With this assumption the following notation makes sense:

$$\Delta_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v.$$

The next axiom can now be stated:

A4. We have the limit

$$\lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon^x(u, v) = \Delta^x(u, v)$$

uniformly with respect to x, u, v in compact set.

Definition 3.3 *A triple (X, d, δ) which satisfies A0, A1, A2, A3, but d^x is degenerate for some $x \in X$, is called degenerate dilatation structure.*

If the triple (X, d, δ) satisfies A0, A1, A2, A3 and d^x is non-degenerate for any $x \in X$, then we call it a weak dilatation structure.

If a weak dilatation structure satisfies A4 then we call it dilatation structure.

3.2 Some induced dilatation structures

Proposition 3.4 *For any $u, v \in U(x)$ let us define*

$$\hat{\delta}^u_\varepsilon v = \Sigma^x_\mu(u, \delta_\varepsilon^{\delta^x_\mu u} \Delta^x_\mu(u, v)) = \delta^x_{\mu^{-1}} \delta_\varepsilon^{\delta^x_\mu u} \delta^x_\mu v.$$

Then $(U(x), \hat{\delta}, (\delta^x, \mu))$ is a dilatation structure.

Proposition 3.5 *With the same notations as in proposition 3.4, the transformation $\Sigma^x_\mu(u, \cdot)$ is an isometry from $(\delta^{\delta^x_\mu u}, \mu)$ to (δ^x, μ). Moreover, we have*

$$\Sigma^x_\mu(u, \delta^x_\mu u) = u.$$

These two propositions show that on a dilatation structure we almost have translations (the infinitesimal sums), which are almost isometries (that is, not with respect to the distance d, but with respect to distances of type (δ^x, μ)). It is almost as if we were working with a conical group, only that we have to use families of distances and to make small shifts in the tangent space (as in the last formula in the proof of propositon 3.4). Moreover, in a very precise way everything converges as $\mu \to 0$ to the right thing. For details see the appendix and [4], [5].

4 Dilatation structures on the boundary of the dyadic tree

Dilatation structures on the boundary of the dyadic tree will have a simpler form than general, mainly because the distance is ultrametric. For the proofs of various results in this section see the corresponding section in [5].

We shall take the group Γ to be the set of integer powers of 2, seen as a subset of dyadic numbers. Thus for any $p \in \mathbb{Z}$ the element $2^p \in \mathbb{Q}_2$ belongs to Γ. The operation is the multiplication of dyadic numbers and the morphism $\nu : \Gamma \to (0, +\infty)$ is defined by

$$\nu(2^p) = d(0, 2^p) = \frac{1}{2^p} \in (0, +\infty) \quad .$$

Axiom A0. This axiom states that for any $p \in \mathbb{N}$ and any $x \in X^\omega$ the dilatation

$$\delta^x_{2^p} : U(x) \to V_{2^p}(x)$$

is a homeomorphism, the sets $U(x)$ and $V_{2^p}(x)$ are open and there is $A > 1$ such that the ball centered in x and radius A is contained in $U(x)$. But this means that $U(x) = X^\omega$, because $X^\omega = B(x, 1)$.

Further, for any $p \in \mathbb{N}$ we have the inclusions:

$$B(x, \frac{1}{2^p}) \subset \delta^x_{2^p} X^\omega \subset V_{2^p}(x) \quad . \tag{4.0.1}$$

For any $p \in \mathbb{N}^*$ the associated dilatation

$$\delta^x_{2^{-p}} : W_{2^{-p}}(x) \rightarrow B(x, B) = X^\omega \quad ,$$

is injective, invertible on the image. We suppose that $W_{2^{-p}}(x)$ is open,

$$V_{2^p}(x) \subset W_{2^{-p}}(x) \tag{4.0.2}$$

and that for all $p \in \mathbb{N}^*$ and $u \in X^\omega$ we have

$$\delta^x_{2^{-p}} \ \delta^x_{2^p} u \ = \ u \ .$$

We leave aside for the moment the interpretation of the technical condition before axiom A4.

Axioms A1 and A2. Nothing simplifies.

Axiom A3. Because d is an ultrametric distance and X^ω is compact, this axiom has very strong consequences, for a non degenerate dilatation structure.

In this case the axiom A3 states that there is a non degenerate distance function d^x on X^ω such that we have the limit

$$\lim_{p\rightarrow\infty} 2^p d(\delta^x_{2^p} u, \delta^x_{2^p} v) = d^x(u, v) \tag{4.0.3}$$

uniformly with respect to $x, u, v \in X^\omega$.

We continue further with properties of weak dilatation structures.

Lemma 4.1 *There exists $p_0 \in \mathbb{N}$ such that for any $x, u, v \in X^\omega$ and for any $p \in \mathbb{N}$, $p \geq p_0$, we have*

$$2^p d(\delta^x_{2^p} u, \delta^x_{2^p} v) = d^x(u, v) \quad .$$

In the sequel p_0 is the smallest natural number satisfying lemma 4.1.

Lemma 4.2 *For any $x \in X^\omega$ and for any $p \in \mathbb{N}$, $p \geq p_0$, we have*

$$\delta^x_{2^p} X^\omega = [x]_p X^\omega \quad .$$

Otherwise stated, for any $x, y \in X^\omega$, any $q \in X^*$, $\mid q \mid \geq p_0$ there exists $w \in X^\omega$ such that

$$\delta^{qx}_{2^{|q|}} w = qy \quad ,$$

and for any $z \in X^\omega$ there is $y \in X^\omega$ such that

$$\delta^{qx}_{2^{|q|}} z = qy \quad .$$

Moreover, for any $x \in X^\omega$ and for any $p \in \mathbb{N}$, $p \geq p_0$ the inclusions from (4.0.1), (4.0.2) are equalities.

The technical condition before the axiom A4 turns out to be trivial. Indeed, from lemma 4.2 it follows that for any $p \geq p_0$, $p \in \mathbb{N}$, and any $x, u, v \in X^\omega$ we have $\delta^x_{2^p} u = [x]_p w$, $w \in X^\omega$. It follows that

$$\delta^x_{2^p} v \in [x]_p X^\omega = W_{2^{-p}}(x) = W_{2^{-p}}(\delta^x_{2^p} u) \quad .$$

Lemma 4.3 *For any* $x, u, v \in X^\omega$ *such that* $2^{p_0} d(x,u) \leq 1$, $2^{p_0} d(x,v) \leq 1$ *we have*

$$d^x(u,v) = d(u,v) \quad .$$

Moreover, under the same hypothesis, for any $p \in \mathbb{N}$ *we have*

$$2^p d(\delta^x_{2^p} u, \delta^x_{2^p} v) = d(u,v) \quad .$$

4.1 Weak dilatation structures on the dyadic tree

The space X^ω decomposes into a disjoint union of 2^{p_0} balls which are isometric. There is no connection between the weak dilatation structures on these balls, therefore we shall suppose further that $p_0 = 0$.

The purpose of this subsection is to find the general form of a weak dilatation structure on X^ω, with $p_0 = 0$.

Definition 4.4 *A function* $W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega)$ *is smooth if for any* $\varepsilon > 0$ *there exists* $\mu(\varepsilon) > 0$ *such that for any* $x, x' \in X^\omega$ *such that* $d(x,x') < \mu(\varepsilon)$ *and for any* $y \in X^\omega$ *we have*

$$\frac{1}{2^k} d(W^x_k(y), W^{x'}_k(y)) \leq \varepsilon \quad ,$$

for an k *such that* $d(x,x') < 1/2^k$.

For the proof of the following theorem see [5]

Theorem 4.5 *Let* (X^ω, d, δ) *be a weak dilatation structure on* (X^ω, d)*, where* d *is the standard distance on* X^ω*, such that* $p_0 = 0$*. Then there exists a smooth (according to definition 4.4) function*

$$W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega) \quad , \quad W(n,x) = W^x_n$$

such that for any $q \in X^*$, $\alpha \in X$, $x, y \in X^\omega$ *we have*

$$\delta^{q\alpha x}_2 q\bar{\alpha} y = q\alpha \bar{x}_1 W^{q\alpha x}_{|q|+1}(y) \quad . \tag{4.1.4}$$

Conversely, to any smooth function $W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega)$ *is associated a weak dilatation structure* (X^ω, d, δ)*, with* $p_0 = 0$*, induced by functions* δ^x_2*, defined by* $\delta^x_2 x = x$ *and otherwise by relation (4.1.4).*

4.2 Self-similar dilatation structures

Let (X^{ω}, d, δ) be a weak dilatation structure. There are induced dilatations structures on $0X^{\omega}$ and $0X^{\omega}$.

Definition 4.6 *For any $\alpha \in X$ and $x, y \in X^{\omega}$ we define $\delta_2^{\alpha,x} y$ by the relation*

$$\delta_2^{\alpha x} \alpha y = \alpha \, \delta_2^{\alpha,x} y \quad .$$

The following proposition has a straightforward proof, therefore we skip it.

Proposition 4.7 *If (X^{ω}, d, δ) is a weak dilatation structure and $\alpha \in X$ then $(X^{\omega}, d, \delta^{\alpha})$ is a weak dilatation structure.*

If (X^{ω}, d, δ') and $(X^{\omega}, d, \delta")$ are weak dilatation structures then (X^{ω}, d, δ) is a weak dilatation structure, where δ is uniquely defined by $\delta^0 = \delta'$, $\delta^1 = \delta"$.

The previous proposition justifies the next definition.

Definition 4.8 *A weak dilatation structure (X^{ω}, d, δ) is self-similar if for any $\alpha \in X$ and $x, y \in X^{\omega}$ we have*

$$\delta_2^{\alpha x} \, \alpha y = \alpha \, \delta_2^{x} y \quad .$$

For the proof of the following proposition see [5].

Proposition 4.9 *Let (X^{ω}, d, δ) be a self-similar weak dilatation structure and $W : \mathbb{N}^* \times X^{\omega} \to Isom(X^{\omega})$ the function associated to it, according to theorem 4.5. Then there exists a function $W : X^{\omega} \to Isom(X^{\omega})$ such that:*

(a) for any $q \in X^*$ and any $x \in X^{\omega}$ we have

$$W^{qx}_{|q|+1} = W^{x} \quad ,$$

(b) there exists $C > 0$ such that for any $x, x', y \in X^{\omega}$ and for any $\lambda > 0$, if $d(x, x') \leq \lambda$ then

$$d(W^{x}(y), W^{x'}(y)) \leq C\lambda \quad .$$

5 The linear group of a dilatation structure

Definition 5.1 *Let (X, d, δ) be a weak dilatation structure. A transformation $A : X \to X$ is linear if it is Lipschitz and it commutes with dilatations in the following sense: for any $x \in X$, $u \in U(x)$ and $\varepsilon \in \Gamma$, $\nu(\varepsilon) < 1$, if $A(u) \in U(A(x))$ then*

$$A \delta^{x}_{\varepsilon} = \delta^{A(x)} A(u) \quad .$$

The group of linear transformations, denoted by $GL(X, d, \delta)$ is formed by all invertible and bi-lipschitz linear transformations of X.

$GL(X,d,\delta)$ is a (local) group. Indeed, we start from the remark that if A is Lipschitz then there exists $C > 0$ such that for all $x \in X$ and $u \in B(x,C)$ we have $A(u) \in U(A(x))$. The inverse of $A \in GL(X,d,\delta)$ is then linear. Same considerations apply for the composition of two linear, bi-lipschitz and invertible transformations.

In the particular case described in the motivation section, namely X normed vector space, d the distance given by the norm, $\Gamma = (0,+\infty)$ and dilatations

$$\delta_\varepsilon^x u = x + \varepsilon(u-x) \quad ,$$

a linear transformations in the sense of definition 5.1 is an affine transformation of the vector space X.

Linear transformations have nice properties which justify the name "linear". We shall use further the (sum and difference) operations

$$\Sigma_\varepsilon^x(u,v) = (\delta_\varepsilon^x)^{-1}\, \delta_\varepsilon^{\delta_\varepsilon^x u} v \quad , \quad \Delta_\varepsilon^x(u,v) = \left(\delta_\varepsilon^{\delta_\varepsilon^x}\right)^{-1} \delta_\varepsilon^x v$$

and the inverse function $inv_\varepsilon^x u = \Delta_\varepsilon^x(u,x)$.

Proposition 5.2 *Let (X,d,δ) be a weak dilatation structure and $A : X \to X$ a linear transformation. Then:*

(a) for all $x \in X$, $u,v \in U(x)$ sufficiently close to x, we have:

$$A\, \Sigma_\varepsilon^x(u,v) = \Sigma_\varepsilon^{A(x)}(A(u),A(v)) \quad .$$

(b) or all $x \in X$, $u \in U(x)$ sufficiently close to x, we have:

$$A\, inv^x(u) = inv^{A(x)} A(u) \quad .$$

(c) for all $x \in X$ the transformation A is derivable and the derivative equals A.

This is important because the sum, difference, inverse operations induced by a dilatation structure give to the space X almost the structure of an affine space. We collect some results from [4] section 4.2, regarding the properties of these operations. Only the last point is new, but with straightforward proof.

Theorem 5.3 *Let (X,d,δ) be a weak dilatation structure. Then, for any $x \in X$, $\varepsilon \in \Gamma$, $\nu(\varepsilon) < 1$, we have:*

(a) for any $u \in U(x)$, $\Sigma_\varepsilon^x(x,u) = u$.

(b) for any $u \in U(x)$ the functions $\Sigma_\varepsilon^x(u,\cdot)$ and $\Delta_\varepsilon^x(u,\cdot)$ are inverse one to another.

(c) the inverse function is shifted involutive: for any $u \in U(x)$,

$$inv_\varepsilon^{\delta_\varepsilon^x u}\, inv_\varepsilon^x(u) = u \quad .$$

(d) the sum operation is shifted associative: for any u, v, w sufficiently close to x we have

$$\Sigma_{\varepsilon}^{x}\left(u, \Sigma_{\varepsilon}^{\delta_{\varepsilon}^{x} u}(v, w)\right) = \Sigma_{\varepsilon}^{x}(\Sigma^{x}(u, v), w) \quad .$$

(e) the difference, inverse and sum operations are related by

$$\Delta_{\varepsilon}^{x}(u, v) = \Sigma_{\varepsilon}^{\delta_{\varepsilon}^{x} u}\left(inv_{\varepsilon}^{x}(u), v\right) \quad ,$$

for any u, v sufficiently close to x.

(f) for any u, v sufficiently close to x and $\mu \in \Gamma$, $\nu(\mu) < 1$, we have:

$$\Delta_{\varepsilon}^{x}\left(\delta_{\mu}^{x} u, \delta_{\mu}^{x} v\right) = \delta_{\mu}^{\delta_{\epsilon\mu}^{x} u} \Delta_{\varepsilon\mu}^{x}(u, v) \quad .$$

Remark that in principle the "translations" $\Sigma_{\varepsilon}^{x}(u, \cdot)$ are not linear. Nevertheless, they commute with dilatation in a known way, according to point (f) theorem 5.3. This is important, because the transformations $\Sigma_{\varepsilon}^{x}(u, \cdot)$ really behave as translations, as explained in subsection 3.2.

The reason for which translations are not linear is that dilatations are not linear. In the case of strong dilatation structures, this happens only when we are in a conical group (see Appendix).

6 A gallery of self-similar groups

In this section we recall the definition of some interesting self-similar groups. In the next section we present linearity results, in terms of dilatation structures, for these groups.

The adding machine. This is the group generated by the transformation a, defined by:

$$\begin{aligned} a(0w) &= 1w \\ a(1w) &= 0a(w) \quad , \end{aligned}$$

where w is an arbitrary infinite word over the alphabet $\{0, 1\}$.

If we identify $\{0, 1\}^{\omega}$ with the dyadic integers $\mathbb{Z}_2$, we have have then

$$a(w) = a(w) + 1 \quad .$$

The adding machine is therefore linear in the classical sense.

The lamplighter group. This is the group generated by the transformations

$$\begin{array}{ll} a(0w) = 1b(w) & b(0w) = 0b(w) \\ a(1w) = 0a(w) & b(1w) = 1a(w) \end{array} \quad .$$

Let us identify X^ω with $(\mathbb{Z}/2\mathbb{Z})[[t]]$ with the help of the function $\Phi : x_1x_2\ldots \mapsto \sum_{i\geq 1} x_i t^{i-1}$. We have then [8]:

$$\Phi(w^{b^{-1}a}) = \Phi(w) + 1, \quad \Phi(w^b) = (1+t)\Phi(w) \quad .$$

This group is therefore linear in the classical sense.

The first Grigorchuk group. The Grigorchuk group [7] is the transformation group of the space $\{0,1\}^\omega$ generated by the transformations a, b, c, d, which are defined by the rules:

$$\begin{array}{ll} a(0w) = 1w & a(1w) = 0w \\ b(0w) = 0a(w) & b(1w) = 1c(w) \\ c(0w) = 0a(w) & c(1w) = 1d(w) \\ d(0w) = 0w & d(1w) = 1b(w) \end{array} \quad .$$

The Grigorchuk group is an infinite finitely generated torsion group. This fact relates it to the General Burnside Problem. This is the first example of a group of intermediate growth [6]. The Tits alternative implies that the Grigorchuk group is not linear in the classical sense.

Some iterated monodromy groups of quadratic rational maps.

The infinite dihedral group. Let a and b be the transformations of the space $X^\omega = \{0,1\}^\omega$, defined by the rules

$$\begin{array}{ll} a(0w) = 1w & b(0w) = 0a(w) \\ a(1w) = 0w & b(1w) = 1b(w) \end{array} \quad .$$

These transformations generate the infinite dihedral group $\mathbb{D}_\infty$. It can be seen as the iterated monodromy group $IMG(z^2)$.

The group $SL(2,\mathbb{Z})$ contains a copy of $\mathbb{D}_\infty$, therefore the infinite dihedral group is linear in the classical sense.

The Basilica group In terms of iterated monodromy groups this is denoted by $IMG(z^2-1)$. It is the group generated by the transformations a, b, which are defined by the rules:

$$\begin{array}{ll} a(0w) = 1b(w) & a(1w) = 0w \\ b(0w) = 0a(w) & b(1w) = 1w \end{array} \quad .$$

This is a group of exponential growth (see ....) . It is a weakly branch group over its commutator (....), therefore by a theorem of Abért ... it is not linear in the classical sense.

The group $IMG(z^2+i)$ This is the group generated by the transformations a, b, c, which are defined by the rules:

$$\begin{array}{ll} a(0w) = 1w & a(1w) = 0w \\ b(0w) = 0a(w) & b(1w) = 1c(w) \\ c(0w) = 0b(w) & c(1w) = 1w \quad . \end{array}$$

It is a group of intermediate growth ... therefore non linear in the classical sense.

Linear groups of self-similar weak dilatation structures Let (X^{ω}, d, δ) be a self-similar dilatation structure on the boundary of the dyadic tree.

Proposition 6.1 *The (local) group* $GL((X^{\omega}, d, \delta)$ *is self-similar.*

Proof. Let $A \in GL(X^{\omega}, d, \delta)$, $\alpha \in X$ and $x, y \in X^{\omega}$. We have then:

$$A\delta_2^{\alpha x} \alpha y = A\left(\alpha \delta_2^x y\right) = A(\alpha) A_{\alpha}\left(\alpha \delta_2^x y\right) \quad .$$

The transformation A is linear, therefore

$$A\delta_2^{\alpha x} \alpha y = \delta_2^{A(\alpha)A_{\alpha}(x)} A(\alpha) A_{\alpha}(y) = A(\alpha) \delta_2^{A_{\alpha}(x)} A_{\alpha}(y) \quad .$$

Compare the right hand sides of the last two lines of equalities. We find that $A_{\alpha} \in GL(X^{\omega}, d, \delta)$. $\square$

7 Linearization results

This is the section of results concerning linearization of some self-similar groups by dilatation structures. RThe next two theorems shows an intriguing fact: at least in the class of groups considered, any linear group (in the classical sense) is also δ-linear for some well chosen dilatation structure on X^{ω}. The other interesting fact will be mentioned after the theorems.

Theorem 7.1 *The first Grigorchuk group is not* δ*-linearizable.*

Theorem 7.2 *In the last column of the following table are listed* δ*-linearity results for some iterated monodromy groups of quadratic rational maps.*

$f(z)$	*comments*	*growth*	*linear*	δ*-linear*
z^2	*Adding machine*	*poly*	*yes*	*yes*
$z^2 - 1$	*Basilica group*	*exp*	*no*	*yes*
$z^2 - 2$	$\mathbb{D}_{\infty}$	*exp*	*yes*	*yes*
$z^2 + i$		*intermediate*	*no*	*no*

Motivated by examples in the previous theorems, we may conjecture that any δ-linearizable group has polynomial or exponential growth.

Question: Is there a Tits alternative for the linear group of a dilatation structure?

8 Appendix

8.1 Groups with dilatations. Conical groups

Metric tangent spaces sometimes have a group structure which is compatible with dilatations. This structure, of a group with dilatations, is interesting by itself. The notion has been introduced in [3]; we describe it further.

Let G be a topological group endowed with an uniformity such that the operation is uniformly continuous. The following description is slightly non canonical, but is nevertheless motivated by the case of a Lie group endowed with a Carnot-Caratheodory distance induced by a left invariant distribution (see for example [2], [3]).

We introduce first the double of G, as the group $G^{(2)} \ = \ G \times G$ with operation

$$(x,u)(y,v) \ = \ (xy, y^{-1}uyv)$$

The operation on the group G, seen as the function

$$op : G^{(2)} \rightarrow G \ , \quad op(x,y) \ = \ xy$$

is a group morphism. Also the inclusions:

$$i' : G \rightarrow G^{(2)} \ , \quad i'(x) \ = \ (x,e)$$

$$i" : G \rightarrow G^{(2)} \ , \quad i"(x) \ = \ (x,x^{-1})$$

are group morphisms.

Definition 8.1 *1. G is an uniform group if we have two uniformity structures, on G and $G \times G$, such that op, i', $i"$ are uniformly continuous.*

2. A local action of a uniform group G on a uniform pointed space (X, x_0) is a function $\phi \in W \in \mathcal{V}(e) \mapsto \hat{\phi} : U_\phi \in \mathcal{V}(x_0) \rightarrow V_\phi \in \mathcal{V}(x_0)$ such that:

   (a) the map $(\phi, x) \mapsto \hat{\phi}(x)$ is uniformly continuous from $G \times X$ (with product uniformity) to X,

   (b) for any $\phi, \psi \in G$ there is $D \in \mathcal{V}(x_0)$ such that for any $x \in D$ $\widehat{\phi\psi^{-1}}(x)$ and $\hat{\phi}(\hat{\psi}^{-1}(x))$ make sense and $\widehat{\phi\psi^{-1}}(x) = \hat{\phi}(\hat{\psi}^{-1}(x))$.

3. Finally, a local group is an uniform space G with an operation defined in a neighbourhood of $(e,e) \subset G \times G$ which satisfies the uniform group axioms locally.

Remark that a local group acts locally at left (and also by conjugation) on itself.

An uniform group, according to the definition (8.1), is a group G such that left translations are uniformly continuous functions and the left action of G on itself is uniformly continuous too. In order to precisely formulate this we need two uniformities: one on G and another on $G \times G$.

These uniformities should be compatible, which is achieved by saying that i', $i"$ are uniformly continuous. The uniformity of the group operation is achieved by saying that the *op* morphism is uniformly continuous.

Definition 8.2 *A group with dilatations (G, δ) is a local uniform group G with a local action of Γ (denoted by δ), on G such that*

H0. the limit $\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$ exists and is uniform with respect to x in a compact neighbourhood of the identity e.

H1. the limit

$$\beta(x, y) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)(\delta_\varepsilon y) \right)$$

is well defined in a compact neighbourhood of e and the limit is uniform.

H2. the following relation holds

$$\lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)^{-1} \right) = x^{-1}$$

where the limit from the left hand side exists in a neighbourhood of e and is uniform with respect to x.

These axioms are in fact a particular version of the axioms for a dilatation structure.

Further we define conical local uniform groups.

Definition 8.3 *A conical group N is a local group with a local action of Γ by morphisms δ_ε such that $\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$ for any x in a neighbourhood of the neutral element e.*

The next proposition explains why a conical group is the infinitesimal version of a group with dilatations.

Proposition 8.4 *Under the hypotheses H0, H1, H2 (G, β, δ) is a conical group, with operation β and dilatations δ.*

Any group with dilatations has an associated dilatation structure on it. In a group with dilatations (G, δ) we define dilatations based in any point $x \in G$ by

$$\delta_\varepsilon^x u = x \delta_\varepsilon (x^{-1} u). \tag{8.1.1}$$

Definition 8.5 *A normed group with dilatations* $(G, \delta, \| \cdot \|)$ *is a group with dilatations* (G, δ) *endowed with a continuous norm function* $\| \cdot \| : G \rightarrow \mathbb{R}$ *which satisfies (locally, in a neighbourhood of the neutral element* e*) the properties:*

(a) for any x *we have* $\|x\| \geq 0$*; if* $\|x\| = 0$ *then* $x = e$*,*

(b) for any x, y *we have* $\|xy\| \leq \|x\| + \|y\|$*,*

(c) for any x *we have* $\|x^{-1}\| = \|x\|$*,*

(d) the limit $\lim_{\varepsilon \rightarrow 0} \frac{1}{\nu(\varepsilon)} \|\delta_\varepsilon x\| = \|x\|^N$ *exists, is uniform with respect to* x *in compact set,*

(e) if $\|x\|^N = 0$ *then* $x = e$*.*

It is easy to see that if $(G, \delta, \| \cdot \|)$ is a normed group with dilatations then $(G, \beta, \delta, \| \cdot \|^N)$ is a normed conical group. The norm $\| \cdot \|^N$ satisfies the stronger form of property (d) definition 8.5: for any $\varepsilon > 0$

$$\|\delta_\varepsilon x\|^N = \varepsilon \|x\|^N .$$

Normed conical groups generalize the notion of Carnot groups.

In a normed group with dilatations we have a natural left invariant distance given by

$$d(x, y) = \|x^{-1} y\| . \tag{8.1.2}$$

Theorem 8.6 *Let* $(G, \delta, \| \cdot \|)$ *be a locally compact normed group with dilatations. Then* (G, δ, d) *is a dilatation structure, where* δ *are the dilatations defined by (8.1.1) and the distance* d *is induced by the norm as in (8.1.2).*

8.2 Tangent bundle of a dilatation structure

Theorem 8.7 *Let* (X, d, δ) *be a weak dilatation structure. Then*

(a) for all $x \in X$*,* $u, v \in X$ *such that* $d(x, u) \leq 1$ *and* $d(x, v) \leq 1$ *and all* $\mu \in (0, A)$ *we have:*

$$d^x(u, v) \ = \ \frac{1}{\mu} d^x(\delta^x_\mu u, \delta^x_\mu v) \ .$$

We shall say that d^x *has the cone property with respect to dilatations.*

(b) we have the following limit:

$$\lim_{\varepsilon \rightarrow 0} \ \frac{1}{\varepsilon} \sup \left\{ \mid d(u, v) - d^x(u, v) \mid \ : \ d(x, u) \leq \varepsilon \ , \ d(x, v) \leq \varepsilon \right\} \ = \ 0 \ .$$

Therefore (X, d) *admits a metric tangent space at* x*, for any point* $x \in X$*.*

For the next theorem we need the previously introduced notion of a conical (local) group.

Theorem 8.8 *Let (X, d, δ) be a dilatation structure. Then for any $x \in X$ the triple $(U(x), \Sigma^x, \delta^x)$ is a conical group. Moreover, left translations of this group are d^x isometries.*

The conical group $(U(x), \Sigma^x, \delta^x)$ can be regarded as the tangent space of (X, d, δ) at x. Further will be denoted by: $T_x X = (U(x), \Sigma^x, \delta^x)$.

The following definition is used in several places.

Definition 8.9 *Let (X, δ, d) be a dilatation structure and $x \in X$ a point. In a neighbourhood $U(x)$ of x, for any $\mu \in (0,1)$ we defined the distances:*

$$(\delta^x, \mu)(u, v) = \frac{1}{\mu} d(\delta^x_\mu u, \delta^x_\mu v).$$

8.3 Strong linear dilatation structures

Theorem 8.10 *Let (X, d, δ) be a weak dilatation structure.*

(a) If dilatations are linear then all transformations $\Delta^x_\varepsilon(u, \cdot)$ are linear for any $u \in X$.

(b) If the dilatation structure is strong then dilatations are linear if and only if the dilatations come from the dilatation structure of a conical group.

Proof. (a) If dilatations are linear, then let $\varepsilon, \mu \in \Gamma$, $\nu(\varepsilon), \nu(\mu) \leq 1$, and $x, y, u, v \in X$ such that the following computations make sense. We have:

$$\Delta^x_\varepsilon(u, \delta^y_\mu v) = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon \delta^y_\mu v \quad .$$

Let $A_\varepsilon = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}}$. We compute:

$$\delta_\mu^{\Delta^x_\varepsilon(u,y)} \Delta^x_\varepsilon(u, v) = \delta_\mu^{A_\varepsilon \delta^x_\varepsilon y} A_\varepsilon \delta^x_\varepsilon v \quad .$$

We use twice the linearity of dilatations:

$$\delta_\mu^{\Delta^x_\varepsilon(u,y)} \Delta^x_\varepsilon(u, v) = A_\varepsilon \delta_\mu^{\delta^x_\varepsilon y} \delta^x_\varepsilon v = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon \delta^y_\mu v \quad .$$

We proved that:

$$\Delta^x_\varepsilon(u, \delta^y_\mu v) = \delta_\mu^{\Delta^x_\varepsilon(u,y)} \Delta^x_\varepsilon(u, v) \quad ,$$

which is the conclusion of the part (a).

(b) Suppose that the dilatation structure is strong. If dilatations are linear, then by point (a) the transformations $\Delta_{\varepsilon}^{x}(u,\cdot)\delta$ are linear for any $u \in X$. Then, with notations made before, for $y = u$ we get

$$\Delta_{\varepsilon}^{x}(u, \delta_{\mu}^{u} v) = \delta_{\mu}^{\delta_{\varepsilon}^{x} u} \Delta_{\varepsilon}^{x}(u,v) \quad ,$$

which implies

$$\delta_{\mu}^{u} v = \Sigma_{\varepsilon}^{x}(u, \delta_{\mu}^{x} \Delta_{\varepsilon}^{x}(u,v)) \quad .$$

We pass to the limit with $\varepsilon \to 0$ and we obtain:

$$\delta_{\mu}^{u} v = \Sigma^{x}(u, \delta_{\mu}^{x} \Delta^{x}(u,v)) \quad .$$

We recognize at the right hand side the dilatations associated to the conical group $T_x X$.

The opposite implication is straightforward, because the dilatation structure of any conical group is linear. $\square$

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