Embeddings of locally finite metric spaces into Banach spaces.

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PART I

Locally finite metric spaces

I Locally finite metric spaces

- Definition of the different embeddings

Definition (lipschitz embedding)

- Let (M, d) and (N, δ) be two metric spaces and an injective map $f : M \to N$. We define the distortion of f to be

$$\begin{aligned} \mathrm{dist}(f) &= \|f\|_{Lip}\|f^{-1}\|_{Lip} \\ &= \sup_{x\neq y\in M} \frac{\delta(f(x),f(y))}{d(x,y)} . \sup_{x\neq y\in M} \frac{d(x,y)}{\delta(f(x),f(y))} . \end{aligned}$$

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- If $\mathrm{dist}(f)$ is finite, we say that f is a lipschitz embedding, or simply an embedding of M into N.
- And if there exists an embedding f from M into N, with $\mathrm{dist}(f) \leq C$, we use the notation $M \overset{C-lip}{\hookrightarrow} N$.

Definition (coarse, uniform and strong uniform embeddings)

- Let (M, d) and (N, δ) be two metric spaces. Suppose $f : M \to N$ is any map and let

$$\varphi_f(t) = \inf\{d(f(x), f(y)) : \ d(x, y) \geq t\} \ t > 0$$

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- so that

$$\varphi_f(d(x, y)) \leq d(f(x), f(y)) \leq \omega_f(d(x, y)) \; \forall x, y \in M.$$

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- On the other hand f is a uniform embedding and M uniformly embeds into N if $\varphi_f(t) > 0$ for all $t > 0$ and $\lim_{t\to 0} \omega_f(t) = 0$.

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- We shall refer to f as a strong uniform embedding if it is both a coarse embedding and a uniform embedding.

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- Questions and partial answers

Question I

Let X be a Banach space. Do we have: ($\forall$ *M separable metric space,* $M \overset{lip}{\hookrightarrow} X) \Rightarrow (c_0 \overset{\simeq}{\hookrightarrow} X)$ *or equivalently* $(c_0 \overset{lip}{\hookrightarrow} X) \Rightarrow (c_0 \overset{\simeq}{\hookrightarrow} X)$.

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[N.J. Kalton]('04)

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[N.J. Kalton]('06)

If c_0 coarsely or uniformly embeds into a Banach space X, then there exists $n \in \mathbb{N}$ s.t. $X^{(n)}$ is non-separable.

Question II

Which metric spaces can be coarsely embedded into a super-reflexive or reflexive Banach space.

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Definition

A metric space M is locally finite if any ball of M with finite radius is finite. If moreover, there is a function $C : (0, +\infty) \to \mathbb{N}$ such that any ball of radius r contains at most $C(r)$ points, we say that M has a bounded geometry.

[N. Brown, E. Guentner]('05)

Let M be a metric space with bounded geometry. There exists a sequence of positive real numbers $\{p_n\}$ and a coarse embedding of M into the ℓ^2- direct sum $\bigoplus \ell^{p_n}(\mathbb{N})$.

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If M is a locally finite metric space then M strongly uniformly embeds into a reflexive Banach space.

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I Locally finite metric spaces

- Definition of the different embeddings
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- Result

Definition

Let X and Y be two Banach spaces. If X and Y are linearly isomorphic, i.e there exists a one-to-one and onto linear application, the *Banach-Mazur distance* between X and Y, denoted by $d_{BM}(X, Y)$, is the infimum of $\|T\| \, \|T^{-1}\|$, over all linear isomorphisms T from X onto Y.

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Definition

For $p \in [1, \infty]$ and $n \in \mathbb{N}$, ℓ_p^n denotes the space $\mathbb{R}^n$ equipped with the ℓ_p norm. We say that a Banach space X uniformly contains the ℓ_p^n's if there is a constant $C \geq 1$ such that for every integer n, X admits an n-dimensional subspace Y so that $d_{BM}(\ell_p^n, Y) \leq C$.

Theorem (F.B., G. Lancien '06)

There exists a universal constant $C > 1$ such that for every Banach space X uniformly containing the ℓ_∞^n's and every locally finite metric space (M, d): $M \overset{C}{\hookrightarrow} X$.

PART II

The hyperbolic tree

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II The hyperbolic tree

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II The hyperbolic tree

- Notation

Denote $\Omega_0 = \{\varnothing\}$, the root of the tree.
Let $\Omega_n = \{-1, 1\}^n$, $T_n = \bigcup_{i=0}^{n} \Omega_i$ and $T = \bigcup_{n=0}^{\infty} T_n$.

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T embeds isometrically into $\ell_1(\mathbb{N})$ in a trivial way. Actually, let $(e_\varepsilon)_{\varepsilon \in T}$ be the canonical basis of $\ell_1(T)$ (T is countable), then the embedding is given by $\varepsilon \mapsto \sum_{s \leq \varepsilon} e_s$.

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[Bourgain]('86)

A Banach space X is not super-reflexive if and only if the finite trees T_n uniformly embed into X (i.e with embedding constants independent of n).

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Theorem (F.B. '06)

Let X be a non super-reflexive Banach space, then (T, ρ) embeds into X.